

# NON-COMMUTATIVE WIDTH AND GOPAKUMAR-VAFA INVARIANTS

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ABSTRACT. We show that the non-commutative widths for flopping curves on smooth 3-folds introduced by Donovan-Wemyss are described by Katz's genus zero Gopakumar-Vafa invariants.

## 1. INTRODUCTION

1.1. **Result.** Let  $X$  be a smooth quasi-projective complex 3-fold and

$$f: X \rightarrow Y$$

a birational flopping contraction which contracts a single rational curve  $\mathbb{P}^1 \cong C \subset X$  to a point  $p \in Y$ . In the paper [DW], Donovan-Wemyss introduced a new invariant associated to  $f$ , the *contraction algebra*  $A_{\text{con}}$ , given by the universal non-commutative deformation algebra of the curve  $C$  in  $X$ . The algebra  $A_{\text{con}}$  is finite dimensional, and it is commutative if and only if  $C$  is not a  $(1, -3)$ -curve. Furthermore if  $A_{\text{con}}$  is commutative, the dimension of  $A_{\text{con}}$  coincides with Reid's width [Rei] of  $C$ . Based on this observation, Donovan-Wemyss defined the following generalizations of Reid's width

$$\text{wid}(C) := \dim_{\mathbb{C}} A_{\text{con}}, \quad \text{cwid}(C) := \dim_{\mathbb{C}} A_{\text{con}}^{\text{ab}}$$

which they called *non-commutative width* and *commutative width* respectively.

On the other hand, Katz [Kat08] defined *genus zero Gopakumar-Vafa (GV) invariants* as virtual numbers of one dimensional stable sheaves on  $X$ . For  $j \geq 1$ , the genus zero GV invariant  $n_j \in \mathbb{Z}_{\geq 0}$  of curve class  $j[C]$  on  $X$  is shown in [Kat08] to coincide with the multiplicity of the Hilbert scheme of  $X$  at some subscheme  $C^{(j)} \subset X$  with curve class  $j[C]$ . The purpose of this short note is to describe Donovan-Wemyss's widths in terms of Katz's genus zero GV invariants. The main result is as follows:

**Theorem 1.1.** *We have the following formulas*

$$(1) \quad \text{wid}(C) = \sum_{j=1}^l j^2 \cdot n_j, \quad \text{cwid}(C) = n_1.$$

Here  $l$  is the scheme theoretic length of  $f^{-1}(p)$  at  $C$ .

Here we remark that the identity of  $\text{cwid}(C)$  is almost obvious from the definitions, and the identity of  $\text{wid}(C)$  is more interesting. The result of Theorem 1.1 indicates that one can study non-commutative widths without using non-commutative algebras<sup>1</sup>. Conversely, one may compute genus zero GV invariants by computing contraction algebras. The proof of Theorem 1.1 is an easy application of the main result of [DW], combined with some deformation argument. By [DW], the algebra  $A_{\text{con}}$  defines the non-commutative twist functor, describing Bridgeland-Chen's flop-flop autoequivalence of  $D^b \text{Coh}(X)$ . On the other hand, after taking the completion at  $p$ , the morphism  $f$  deforms to flopping contractions of disjoint  $(-1, -1)$ -curves, such that the number of  $(-1, -1)$ -curves with curve class  $j[C]$  coincides with  $n_j$ . Now the flop-flop autoequivalence deforms along the deformation of  $f$ , hence the non-commutative twist functor also deforms: the resulting deformation is a composition of Seidel-Thomas's spherical twists along  $(-1, -1)$ -curves. We then relate the Hilbert polynomial of a cohomology sheaf of the kernel object of the non-commutative twist functor with that of the above composition of the spherical twists, and obtain the desired identity of  $\text{wid}(C)$ .

**1.2. Examples and a Remark.** Here we describe some examples of Theorem 1.1.

**Example 1.2.** *In Theorem 1.1, we have  $l = 1$  if and only if  $C$  is either a  $(-1, -1)$  or a  $(0, -2)$ -curve. In this case, we have  $\text{wid}(C) = \text{cwid}(C)$ , and it coincides with Reid's width (cf. [DW, Example 3.12]). On the other hand, the genus zero GV invariant  $n_1$  also coincides with Reid's width as indicated in [BKL01, Section 1].*

**Example 1.3.** *Suppose that  $Y = \text{Spec } R_k$ , where  $R_k$  is defined by*

$$R_k = \mathbb{C}[u, v, x, y]/(u^2 + v^2y = x(x^2 + y^{2k+1})).$$

*There is a flopping contraction  $f: X \rightarrow Y$  with  $l = 2$ . The contraction algebra  $A_{\text{con}}$  is computed in [DW, Example 3.14]*

$$\begin{aligned} A_{\text{con}} &\cong \mathbb{C}\langle x, y \rangle / (xy = -yx, x^2 = y^{2k+1}) \\ A_{\text{con}}^{\text{ab}} &\cong \mathbb{C}[x, y] / (xy = 0, x^2 = y^{2k+1}). \end{aligned}$$

*It follows that*

$$\text{wid}(C) = 3(2k + 1), \quad \text{cwid}(C) = 2k + 3.$$

*The result of Theorem 1.1 indicates that  $n_1 = 2k + 3$  and  $n_2 = k$ .*

We also have the following remark:

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<sup>1</sup>Wemyss pointed out to the author that the non-commutative widths are commutative things, as they are computed using some Ext-groups on commutative algebras. See [DW, Remark 5.2].

**Remark 1.4.** *We have  $n_j \geq 1$  for  $1 \leq j \leq l$ . So Theorem 1.1 implies that*

$$\text{wid}(C) \geq \sum_{j=1}^l j^2.$$

*The above lower bound is better than the lower bound in [DW, Remark 3.17].*

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## 2. PRELIMINARY

**2.1. 3-fold flopping contractions.** Let  $X$  be a smooth quasi-projective complex 3-fold. By definition, a *flopping contraction* is a birational morphism

$$(2) \quad f: X \rightarrow Y$$

which is isomorphic in codimension one,  $Y$  has only Gorenstein singularities and the relative Picard number of  $f$  equals to one. In what follows, we always assume that the exceptional locus  $C$  of  $f$  is isomorphic to  $\mathbb{P}^1$ , and set

$$p := f(C) \in Y.$$

We say that  $C \subset X$  is  $(a, b)$  curve if  $N_{C/X}$  is isomorphic to  $\mathcal{O}_C(a) \oplus \mathcal{O}_C(b)$ . It is well-known that  $(a, b)$  is either one of the following:

$$(a, b) = (-1, -1), (0, -2), (1, -3).$$

We denote by  $l$  the length of  $\mathcal{O}_{f^{-1}(p)}$  at the generic point of  $C$ , where  $f^{-1}(p)$  is the scheme theoretic fiber of  $f$  at  $p$ . Then we have

$$l \in \{1, 2, 3, 4, 5, 6\}$$

and  $l = 1$  if and only if  $C$  is not a  $(1, -3)$ -curve (cf. [KM92, Section 1]). Moreover if  $l = 1$ , then we have

$$(3) \quad \widehat{\mathcal{O}}_{Y,p} \cong \mathbb{C}[[x, y, z, w]] / (x^2 + y^2 + z^2 + w^{2k})$$

for some  $k \in \mathbb{Z}_{\geq 1}$ . The number  $k$  is called *width* of  $C$  in [Rei].

**2.2. Contraction algebras.** In the setting of Subsection 2.1, we set  $R = \widehat{\mathcal{O}}_{Y,p}$ , and take the following completion of (2)

$$(4) \quad \widehat{f}: \widehat{X} := X \times_Y \operatorname{Spec} R \rightarrow \widehat{Y} := \operatorname{Spec} R.$$

Then there is a line bundle  $\mathcal{L}$  on  $\widehat{X}$  such that  $\deg(\mathcal{L}|_C) = 1$ . We define the vector bundle  $\mathcal{N}$  on  $\widehat{X}$  to be the extension

$$0 \rightarrow \mathcal{L}^{-1} \rightarrow \mathcal{N} \rightarrow \mathcal{O}_{\widehat{X}}^{\oplus r} \rightarrow 0$$

given by the minimum generators of  $H^1(\widehat{X}, \mathcal{L}^{-1})$ . We set  $\mathcal{U} := \mathcal{O}_{\widehat{X}} \oplus \mathcal{N}$ ,  $N := \widehat{f}_* \mathcal{N}$  and

$$A := \operatorname{End}_{\widehat{X}}(\mathcal{U}) \cong \operatorname{End}_R(R \oplus N).$$

By Van den Bergh [dB04, Section 3.2.8], we have a derived equivalence

$$(5) \quad \mathbf{R} \operatorname{Hom}_{\widehat{X}}(\mathcal{U}, -): D^b \operatorname{Coh}(\widehat{X}) \xrightarrow{\sim} D^b \operatorname{mod} A$$

whose inverse is given by  $-\overset{\mathbf{L}}{\otimes}_A \mathcal{U}$ . Here  $\operatorname{mod} A$  is the category of finitely generated right  $A$ -modules.

**Definition 2.1.** ([DW, Definition 2.11]) *The contraction algebra  $A_{\operatorname{con}}$  is defined to be  $A/I_{\operatorname{con}}$ , where  $I_{\operatorname{con}}$  is the two sided ideal of  $A$  consisting of morphisms  $R \oplus N \rightarrow R \oplus N$  factoring through a member of  $\operatorname{add}(R)$ . Here  $\operatorname{add}(R)$  is the set of summands of finite sums of  $R$ .*

By [DW, Proposition 2.12], the algebra  $A_{\operatorname{con}}$  is finite dimensional.

**Remark 2.2.** *The algebra  $A_{\operatorname{con}}$  is commutative if and only if  $C$  is not a  $(1, -3)$ -curve (cf. [DW, Theorem 3.15]). In this case,  $A_{\operatorname{con}}$  is isomorphic to  $\mathbb{C}[t]/(t^k)$ , where  $k$  is the width of  $C$  which appears in (3). See [DW, Example 3.12].*

The contraction algebra  $A_{\operatorname{con}}$  coincides with the universal algebra which represents the non-commutative deformation functor of  $\mathcal{O}_C(-1)$

$$(6) \quad \operatorname{Def}_{\mathcal{O}_C(-1)}: \operatorname{Art}_1 \rightarrow \operatorname{Sets}.$$

Here  $\operatorname{Art}_1$  is the category of finite dimensional  $\mathbb{C}$ -algebras  $\Gamma$  with some additional conditions, and the functor (6) assigns each  $\Gamma$  to the set of isomorphism classes of flat deformation of  $\mathcal{O}_C(-1)$  to  $\operatorname{Coh}(\mathcal{O}_X \otimes_{\mathbb{C}} \Gamma)$ . We refer [DW, Section 2] for details of the functor (6). Since  $A_{\operatorname{con}}$  represents the functor (6), there is the universal non-commutative deformation of  $\mathcal{O}_C(-1)$

$$(7) \quad \mathcal{E} \in \operatorname{Coh}(\mathcal{O}_X \otimes_{\mathbb{C}} A_{\operatorname{con}}).$$

Let  $A_{\operatorname{con}}^{\operatorname{ab}}$  be the abelization of  $A_{\operatorname{con}}$ . The algebra  $A_{\operatorname{con}}^{\operatorname{ab}}$  is a commutative Artinian local  $\mathbb{C}$ -algebra, which represents the commutative deformation functor

$$(8) \quad \operatorname{cDef}_{\mathcal{O}_C(-1)}: \operatorname{cArt}_1 \rightarrow \operatorname{Sets}.$$

Here  $\text{cArt}_1$  is the category of commutative Artinian local  $\mathbb{C}$ -algebras, and the functor (8) is the restriction of the functor (6) to  $\text{cArt}_1$ . We refer [DW, Section 3] for details of the above representabilities.

**2.3. Flop equivalences.** The contraction algebra  $A_{\text{con}}$  plays an important role in describing Bridgeland-Chen's flop-flop autoequivalence. Let us consider the flop diagram of (2)

$$(9) \quad \begin{array}{ccc} X & \xrightarrow{\phi} & X^\dagger \\ & \searrow f & \swarrow f^\dagger \\ & Y & \end{array}$$

By [Bri02] and [Che02], we have the derived equivalence

$$(10) \quad \Phi_{X \rightarrow X^\dagger}^{\mathcal{O}_{X \times_Y X^\dagger}} : D^b \text{Coh}(X) \xrightarrow{\sim} D^b \text{Coh}(X^\dagger).$$

Here we use the notation in Appendix A for the Fourier-Mukai functors. Composing (10) twice, we obtain the autoequivalence

$$(11) \quad \Phi_{X^\dagger \rightarrow X}^{\mathcal{O}_{X \times_Y X^\dagger}} \circ \Phi_{X \rightarrow X^\dagger}^{\mathcal{O}_{X \times_Y X^\dagger}} : D^b \text{Coh}(X) \xrightarrow{\sim} D^b \text{Coh}(X).$$

The result of [DW, Proposition 7.18] shows that (11) has an inverse isomorphic to the non-commutative twist functor  $T_{\mathcal{E}}$  associated to the universal object (7). Namely  $T_{\mathcal{E}}$  is the autoequivalence of  $D^b \text{Coh}(X)$  which fits into the distinguished triangle

$$(12) \quad \mathbf{R} \text{Hom}(\mathcal{E}, F) \stackrel{\mathbf{L}}{\otimes}_{A_{\text{con}}} \mathcal{E} \rightarrow F \rightarrow T_{\mathcal{E}}(F)$$

for any  $F \in D^b \text{Coh}(X)$ . If  $C$  is a  $(-1, -1)$ -curve, the functor  $T_{\mathcal{E}}$  coincides with Seidel-Thomas twist [ST01] along  $\mathcal{O}_C(-1)$ . If  $C$  is a  $(0, -2)$ -curve, then  $T_{\mathcal{E}}$  coincides with the author's generalized twist [Tod07]<sup>2</sup>. The kernel object of the equivalence  $T_{\mathcal{E}}$  is given by

$$\text{Cone} \left( \mathbf{R} \text{Hom}_A(A_{\text{con}}, A) \stackrel{\mathbf{L}}{\otimes}_{A^{\text{op}} \otimes A} (\mathcal{U}^\vee \boxtimes \mathcal{U}) \rightarrow \mathcal{O}_{\Delta_X} \right).$$

Here  $\Delta_X \subset X \times X$  is the diagonal (cf. [DW, Lemma 6.16]).

**Lemma 2.3.** *The object  $\mathbf{R} \text{Hom}_A(A_{\text{con}}, A) \stackrel{\mathbf{L}}{\otimes}_{A^{\text{op}} \otimes A} (\mathcal{U}^\vee \boxtimes \mathcal{U})$  is isomorphic to  $\mathcal{F}[-2]$  for  $\mathcal{F} \in \text{Coh}(X \times X)$  satisfying the following: there is a filtration*

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_{\dim A_{\text{con}}} = \mathcal{F}$$

*such that each subquotient  $\mathcal{F}_j/\mathcal{F}_{j-1}$  is isomorphic to  $\mathcal{O}_C(-1) \boxtimes \mathcal{O}_C(-1)$ .*

<sup>2</sup>In [Tod07], it was stated that  $T_{\mathcal{E}}$  is isomorphic to (11), but it was wrong: the correct statement is  $T_{\mathcal{E}}$  is an inverse of (11). We explain details in Appendix B.

*Proof.* By the definition of  $\text{Art}_1$  in [DW, Definition 2.1], there is a  $\mathbb{C}$ -algebra homomorphism  $A_{\text{con}} \rightarrow \mathbb{C}$  such that its kernel  $\mathfrak{n} \subset A_{\text{con}}$  is nilpotent. The ideal  $\mathfrak{n} \subset A_{\text{con}}$  is two-sided, and  $A_{\text{con}}/\mathfrak{n}$  is a one dimensional  $A^{\text{op}} \otimes A$ -module. We have the filtration of  $A^{\text{op}} \otimes A$ -modules

$$0 = \mathfrak{n}^m \subset \mathfrak{n}^{m-1} \subset \cdots \subset \mathfrak{n} \subset A_{\text{con}}$$

for some  $m > 0$  such that each subquotient  $\mathfrak{n}^i/\mathfrak{n}^{i+1}$  is an  $A_{\text{con}}/\mathfrak{n}$ -module. Since  $A_{\text{con}}/\mathfrak{n} = \mathbb{C}$ , the object  $\mathfrak{n}^i/\mathfrak{n}^{i+1}$  is a finite direct sum of  $A_{\text{con}}/\mathfrak{n}$ . Therefore it is enough to show that

$$(13) \quad \begin{aligned} \mathbf{R} \text{Hom}_A(A_{\text{con}}/\mathfrak{n}, A) \otimes_{A^{\text{op}} \otimes A}^{\mathbf{L}} (\mathcal{U}^\vee \boxtimes \mathcal{U}) \\ \cong \mathcal{O}_C(-1) \boxtimes \mathcal{O}_C(-1)[-2]. \end{aligned}$$

Let  $S \in \text{mod } A$  be the object given by

$$S := \mathbf{R} \text{Hom}_{\widehat{X}}(\mathcal{U}, \mathcal{O}_C(-1)).$$

Note that we have  $\dim_{\mathbb{C}} S = 1$ . The object  $S$  is the unique simple  $A_{\text{con}}$ -module (cf. [DW, Section 2.3]), hence  $A_{\text{con}}/\mathfrak{n}$  viewed as a right  $A_{\text{con}}$ -module is isomorphic to  $S$ . On the other hand, the vector bundle  $\mathcal{U}^\vee \boxtimes \mathcal{U}$  on  $\widehat{X} \times \widehat{X}$  is a tilting vector bundle. Hence we have a derived equivalence

$$\mathbf{R} \text{Hom}_{\widehat{X} \times \widehat{X}}(\mathcal{U}^\vee \boxtimes \mathcal{U}, -): D^b \text{Coh}(\widehat{X} \times \widehat{X}) \xrightarrow{\sim} D^b \text{mod}(A^{\text{op}} \otimes A)$$

with inverse given by  $-\otimes_{A^{\text{op}} \otimes A}^{\mathbf{L}} (\mathcal{U}^\vee \boxtimes \mathcal{U})$ . Let  $\mathbb{D}$  be the dualizing functor  $\mathbf{R} \mathcal{H}om_{\widehat{X}}(-, \mathcal{O}_{\widehat{X}})$  on  $D^b \text{Coh}(\widehat{X})$ . We have  $\mathbb{D}(\mathcal{O}_C(-1)) \cong \mathcal{O}_C(-1)[-2]$ , and

$$\begin{aligned} & \mathbf{R} \text{Hom}_{\widehat{X} \times \widehat{X}}(\mathcal{U}^\vee \boxtimes \mathcal{U}, \mathcal{O}_C(-1) \boxtimes \mathcal{O}_C(-1)[-2]) \\ & \cong \mathbf{R} \text{Hom}_{\widehat{X} \times \widehat{X}}(\mathbb{D}(\mathcal{U}) \boxtimes \mathcal{U}, \mathbb{D}(\mathcal{O}_C(-1)) \boxtimes \mathcal{O}_C(-1)) \\ & \cong \mathbf{R} \text{Hom}_{\widehat{X}}(\mathcal{O}_C(-1), \mathcal{U}) \otimes_{\mathbb{C}} \mathbf{R} \text{Hom}_{\widehat{X}}(\mathcal{U}, \mathcal{O}_C(-1)) \\ & \cong \mathbf{R} \text{Hom}_A(S, A) \otimes_{\mathbb{C}} S \\ & \cong \mathbf{R} \text{Hom}_A(A_{\text{con}}/\mathfrak{n}, A). \end{aligned}$$

Therefore we obtain the desired isomorphism (13).  $\square$

**2.4. Genus zero Gopakumar-Vafa invariants.** The genus zero GV invariants defined in [Kat08] count one dimensional stable sheaves  $F$  on Calabi-Yau 3-folds satisfying  $\chi(F) = 1$ . In the setting of Subsection 2.1, the variety  $X$  may not be Calabi-Yau, but so in a neighborhood of  $C$ . Since  $C$  is rigid in  $X$ , we can define the genus zero GV invariant with curve class  $j[C]$  on  $X$  as well. Indeed in [Kat08], the genus zero GV invariants of  $X$  are shown to coincide with the multiplicities of the Hilbert scheme of  $X$  at some subschemes supported on  $C$ . Let  $p \in H \subset Y$  be a general hypersurface, and  $\overline{H} \subset X$  its proper

transform. Then we have  $C \subset \overline{H}$ . Let  $I \subset \mathcal{O}_{\overline{H}}$  be the ideal sheaf of  $C$ . For  $j \geq 1$ , we have the subscheme  $C^{(j)} \subset X$  given by

$$\mathcal{O}_{C^{(j)}} = (\mathcal{O}_{\overline{H}}/I^j)/Q$$

where  $Q$  is the maximum zero dimensional subsheaf of  $\mathcal{O}_{\overline{H}}/I^j$ . Let  $\text{Hilb}(X)$  be the Hilbert scheme parameterizing closed subschemes in  $X$ . If  $1 \leq j \leq l$ , it is shown in [BKL01, Section 2.1], that  $C^{(j)}$  is the isolated point in  $\text{Hilb}(X)$ , and the following number is defined:

**Definition 2.4.** For  $1 \leq j \leq l$ , we define  $n_j \in \mathbb{Z}_{\geq 1}$  to be

$$n_j := \dim_{\mathbb{C}} \mathcal{O}_{\text{Hilb}(X), C^{(j)}}.$$

By convention, we define  $n_j = 0$  for  $j > l$ .

Since  $\mathcal{O}_{\text{Hilb}(X), C^{(j)}}$  is a finitely generated Artinian  $\mathbb{C}$ -algebra, the number  $n_j$  is well-defined. If  $l = 1$ , the number  $n_1$  equals to the width  $k$  in (3) as indicated in [BKL01, Section 1]. In general, Katz [Kat08] shows that  $n_j$  coincides with the genus zero GV invariant of  $X$  with curve class  $j[C]$ .

The number  $n_j$  also appears in the context of deformations in the following way. By [BKL01, Section 2.1], there exists a flat deformation of (4)

$$(14) \quad \begin{array}{ccc} \mathcal{X} & \xrightarrow{g} & \mathcal{Y} \\ & \searrow & \downarrow \\ & & T \end{array}$$

where  $T$  is a Zariski open neighborhood of  $0 \in \mathbb{A}^1$  such that  $g_0: \mathcal{X}_0 \rightarrow \mathcal{Y}_0$  is isomorphic to  $\hat{f}$  in (4), and  $g_t: \mathcal{X}_t \rightarrow \mathcal{Y}_t$  for  $t \in T \setminus \{0\}$  is a flopping contraction whose exceptional locus is a disjoint union of  $(-1, -1)$ -curves. Here  $\mathcal{X}_t, \mathcal{Y}_t$  are the fibers of  $\mathcal{X} \rightarrow T, \mathcal{Y} \rightarrow T$  at  $t \in T$  respectively. Then the number  $n_j$  coincides with the number of  $g_t$ -exceptional  $(-1, -1)$ -curves  $C' \subset \mathcal{X}_t$  for  $t \neq 0$  whose curve class equals to  $j[C]$ , i.e. for any line bundle  $\mathcal{L}$  on  $\mathcal{X}$ , we have

$$(15) \quad \deg(\mathcal{L}|_{C'}) = j \deg(\mathcal{L}|_C)$$

where we regard  $C$  as a curve on the central fiber of  $\mathcal{X} \rightarrow T$ . In what follows, we write the exceptional locus of  $g_t$  for  $t \neq 0$  as

$$C_{j,k} \subset \mathcal{X}_t, \quad 1 \leq j \leq l, \quad 1 \leq k \leq n_j$$

where  $C_{j,k}$  is a  $(-1, -1)$ -curve with curve class  $j[C]$ .

### 3. PROOF OF THEOREM 1.1

*Proof.* The identity  $\text{cwid}(C) = n_1$  is almost obvious from the definitions of both sides. Indeed since  $A_{\text{con}}^{\text{ab}}$  represents the commutative deformation functor (8), the scheme  $\text{Spec } A_{\text{con}}^{\text{ab}}$  is the component of the moduli scheme of one dimensional stable sheaves on  $X$  containing

$\mathcal{O}_C(-1)$ . By tensoring the line bundle  $\mathcal{L}$  in Subsection 2.2, the scheme  $\mathrm{Spec} A_{\mathrm{con}}^{\mathrm{ab}}$  is isomorphic to the component of the moduli scheme of stable sheaves containing  $\mathcal{O}_C$ , which defines the invariant  $n_1$ . By the proof of [Kat08, Proposition 3.3], the degree of the virtual fundamental cycle of  $\mathrm{Spec} A_{\mathrm{con}}^{\mathrm{ab}}$  coincides with the dimension of  $A_{\mathrm{con}}^{\mathrm{ab}}$ . Therefore  $\mathrm{cwid}(C) = n_1$  holds.

We show the identity of  $\mathrm{wid}(C)$ . The morphism  $g$  in (14) is a flopping contraction, and the argument of [Che02, Section 6] shows that  $g$  admits a flop

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\psi} & \mathcal{X}^\dagger \\ & \searrow g & \swarrow g^\dagger \\ & \mathcal{Y} & \end{array}$$

such that we have the derived equivalence

$$\Phi_{\mathcal{X} \rightarrow \mathcal{X}^\dagger}^{\mathcal{O}_{\mathcal{X} \times \mathcal{Y} \mathcal{X}^\dagger}} : D^b \mathrm{Coh}(\mathcal{X}) \xrightarrow{\sim} D^b \mathrm{Coh}(\mathcal{X}^\dagger).$$

By composing the above equivalence twice, we obtain the autoequivalence

$$(16) \quad \Phi_{\mathcal{X}^\dagger \rightarrow \mathcal{X}}^{\mathcal{O}_{\mathcal{X} \times \mathcal{Y} \mathcal{X}^\dagger}} \circ \Phi_{\mathcal{X} \rightarrow \mathcal{X}^\dagger}^{\mathcal{O}_{\mathcal{X} \times \mathcal{Y} \mathcal{X}^\dagger}} : D^b \mathrm{Coh}(\mathcal{X}) \xrightarrow{\sim} D^b \mathrm{Coh}(\mathcal{X}).$$

Let  $\Psi$  be an inverse of the equivalence (16), and

$$\mathcal{P} \in D^b \mathrm{Coh}(\mathcal{X} \times_T \mathcal{X})$$

the kernel object of  $\Psi$ . By [Che02, Lemma 6.1], for each  $t \in T$ , we have the commutative diagram

$$\begin{array}{ccc} D^b \mathrm{Coh}(\mathcal{X}) & \xrightarrow{\Psi} & D^b \mathrm{Coh}(\mathcal{X}) \\ \downarrow \mathbf{L}i_t^* & & \downarrow \mathbf{L}i_t^* \\ D^b \mathrm{Coh}(\mathcal{X}_t) & \xrightarrow{\Psi_t} & D^b \mathrm{Coh}(\mathcal{X}_t). \end{array}$$

Here  $i_t: \mathcal{X}_t \hookrightarrow \mathcal{X}$  is the inclusion, and  $\Psi_t$  is the Fourier-Mukai functor with kernel  $\mathcal{P}_t := \mathbf{L}j_t^* \mathcal{P}$ , where  $j_t$  is the inclusion

$$j_t := (i_t \times i_t): \mathcal{X}_t \times \mathcal{X}_t \hookrightarrow \mathcal{X} \times_T \mathcal{X}.$$

The functor  $\Psi_t$  is an equivalence, and it has an inverse given by the composition (cf. [Che02, Corollary 4.5])

$$(17) \quad \Phi_{\mathcal{X}_t^\dagger \rightarrow \mathcal{X}_t}^{\mathcal{O}_{\mathcal{X}_t \times \mathcal{Y}_t \mathcal{X}_t^\dagger}} \circ \Phi_{\mathcal{X}_t \rightarrow \mathcal{X}_t^\dagger}^{\mathcal{O}_{\mathcal{X}_t \times \mathcal{Y}_t \mathcal{X}_t^\dagger}} : D^b \mathrm{Coh}(\mathcal{X}_t) \xrightarrow{\sim} D^b \mathrm{Coh}(\mathcal{X}_t).$$

Therefore by [DW, Proposition 7.18], the equivalence  $\Psi_0$  is isomorphic to the non-commutative twist functor  $T_{\mathcal{E}}$  in (12). By the uniqueness of Fourier-Mukai kernels in Lemma A.1 below, we have

$$(18) \quad \mathcal{P}_0 \cong \mathrm{Cone}(\mathcal{F}_0[-2] \rightarrow \mathcal{O}_{\Delta_{\mathcal{X}_0}}).$$

Here  $\mathcal{F}_0$  is a sheaf  $\mathcal{F}$  on  $X \times X$  given in Lemma 2.3, restricted to  $\widehat{X} \times \widehat{X}$ .



For  $t \neq 0$ , the birational map  $\mathcal{X}_t \dashrightarrow \mathcal{X}_t^\dagger$  is the composition of flops at  $(-1, -1)$ -curves  $C_{j,k}$  for  $1 \leq j \leq l$ ,  $1 \leq k \leq n_j$ . Hence the equivalence  $\Psi_t$  for  $t \neq 0$  is isomorphic to the compositions of all the spherical twists along  $\mathcal{O}_{C_{j,k}}(-1)$  for  $1 \leq j \leq l$ ,  $1 \leq k \leq n_j$ . Therefore using Lemma A.1 again, we have

$$(19) \quad \mathcal{P}_t \cong \text{Cone}(\mathcal{F}_t[-2] \rightarrow \mathcal{O}_{\Delta_{\mathcal{X}_t}})$$

where  $\mathcal{F}_t$  is a sheaf on  $\mathcal{X}_t \times \mathcal{X}_t$  defined by

$$(20) \quad \mathcal{F}_t := \bigoplus_{j=1}^l \bigoplus_{k=1}^{n_j} \mathcal{O}_{C_{j,k}}(-1) \boxtimes \mathcal{O}_{C_{j,k}}(-1).$$

**Lemma 3.1.** *We have  $\mathcal{H}^i(\mathcal{P}) = 0$  for  $i \neq 0, 1$ .*

*Proof.* For any  $t \in T$ , we have the distinguished triangle

$$\mathcal{P} \rightarrow \mathcal{P} \rightarrow j_{t*}\mathcal{P}_t.$$

By (18) and (19), we have  $\mathcal{H}^i(\mathcal{P}_t) = 0$  for any  $t \in T$  and  $i \neq 0, 1$ . By taking the long exact sequence of cohomologies of the above triangle, we obtain  $j_t^*\mathcal{H}^i(\mathcal{P}) = 0$  for any  $t \in T$  and  $i \neq 0, 1$ . Therefore we have  $\mathcal{H}^i(\mathcal{P}) = 0$  for  $i \neq 0, 1$ .  $\square$

**Lemma 3.2.** *We have  $\mathcal{H}^0(\mathcal{P}) \cong \mathcal{O}_{\Delta_{\mathcal{X}}}$  and  $\mathcal{H}^1(\mathcal{P})$  is flat over  $T$ . Furthermore we have  $j_t^*\mathcal{H}^1(\mathcal{P}) \cong \mathcal{F}_t$  for any  $t \in T$ .*

*Proof.* By Lemma 3.1, we have the distinguished triangle in  $D^b \text{Coh}(\mathcal{X} \times_T \mathcal{X})$

$$\mathcal{H}^0(\mathcal{P}) \rightarrow \mathcal{P} \rightarrow \mathcal{H}^1(\mathcal{P})[-1].$$

Applying  $\mathbf{L}j_t^*$ , we obtain the distinguished triangle in  $D^b \text{Coh}(\mathcal{X}_t \times \mathcal{X}_t)$

$$\mathbf{L}j_t^*\mathcal{H}^0(\mathcal{P}) \rightarrow \mathcal{P}_t \rightarrow \mathbf{L}j_t^*\mathcal{H}^1(\mathcal{P})[-1].$$

By taking the long exact sequence of cohomologies, we have

$$\mathbf{L}j_t^*\mathcal{H}^0(\mathcal{P}) \cong j_t^*\mathcal{H}^0(\mathcal{P}), \quad \mathcal{F}_t \cong j_t^*\mathcal{H}^1(\mathcal{P})$$

and the exact sequence

$$(21) \quad 0 \rightarrow j_t^*\mathcal{H}^0(\mathcal{P}) \rightarrow \mathcal{O}_{\Delta_{\mathcal{X}_t}} \rightarrow \mathcal{H}^{-1}(\mathbf{L}j_t^*\mathcal{H}^1(\mathcal{P})) \rightarrow 0.$$

Below we denote by  $\Delta: \mathcal{X} \rightarrow \mathcal{X} \times_T \mathcal{X}$ ,  $\Delta_t: \mathcal{X}_t \rightarrow \mathcal{X}_t \times \mathcal{X}_t$  the diagonal morphisms, and  $\mathcal{C}$  the exceptional locus of  $g: \mathcal{X} \rightarrow \mathcal{Y}$ . The isomorphism  $\mathcal{F}_t \cong j_t^*\mathcal{H}^1(\mathcal{P})$  implies that  $\mathcal{H}^1(\mathcal{P})$  is supported on  $\mathcal{C} \times \mathcal{C}$ , hence  $\mathcal{H}^{-1}(\mathbf{L}j_t^*\mathcal{H}^1(\mathcal{P}))$  is supported on  $\mathcal{C}_t \times \mathcal{C}_t$ . The exact sequence (21) also implies that  $\mathcal{H}^{-1}(\mathbf{L}j_t^*\mathcal{H}^1(\mathcal{P}))$  is supported on  $\Delta_{\mathcal{X}_t}$ , hence on  $\Delta_{\mathcal{X}_t} \cap (\mathcal{C}_t \times \mathcal{C}_t) = \Delta_t(\mathcal{C}_t)$ . It follows that  $\mathcal{H}^0(\mathcal{P})$  is written as  $\Delta_*\mathcal{I}$  for a rank one torsion free sheaf  $\mathcal{I}$  on  $\mathcal{X}$ , and the exact sequence (21) is given by  $\Delta_{t*}$  of the exact sequence of the following form

$$(22) \quad 0 \rightarrow i_t^*\mathcal{I} \rightarrow \mathcal{O}_{\mathcal{X}_t} \rightarrow \mathcal{O}_{\mathcal{C}'_t} \rightarrow 0$$

for some subscheme  $\mathcal{C}'_t \subset \mathcal{X}_t$  supported on  $\mathcal{C}_t$ . Also by the generic flatness, there is a non-empty Zariski open subset  $U \subset T$  such that  $\mathcal{H}^{-1}(\mathbf{L}j_t^* \mathcal{H}^1(\mathcal{P})) = 0$  for all  $t \in U$ . This implies that  $\mathcal{C}'_t = \emptyset$  for all  $t \in U$ , hence  $\mathcal{I}$  is isomorphic to  $\mathcal{O}_{\mathcal{X}}$  away from  $\mathcal{C}_t$  for  $t \in T \setminus U$ . By taking the double dual of  $\mathcal{I}$ , we obtain the exact sequence

$$0 \rightarrow \mathcal{I} \rightarrow \mathcal{O}_{\mathcal{X}} \rightarrow \mathcal{O}_{\mathcal{C}'} \rightarrow 0$$

where  $\mathcal{C}'$  is supported on  $\mathcal{C}_t$  for  $t \in T \setminus U$ . If  $\mathcal{C}' \neq \emptyset$ , then  $j_t^* \mathcal{H}^0(\mathcal{P}) \cong \Delta_{t*} i_t^* \mathcal{I}$  contains the non-zero sheaf  $\Delta_{t*} \mathcal{H}^{-1} \mathbf{L}i_t^*(\mathcal{O}_{\mathcal{C}'})$  for some  $t \in T \setminus U$  supported on  $\mathcal{C}_t$ , which contradicts to (21). Therefore  $\mathcal{C}' = \emptyset$  and  $\mathcal{H}^0(\mathcal{P}) \cong \mathcal{O}_{\Delta_{\mathcal{X}}}$  holds.

Now in the sequence (22), we have  $i_t^* \mathcal{I} \cong \mathcal{O}_{\mathcal{X}_t}$  for any  $t \in T$ , hence  $\mathcal{C}'_t = \emptyset$  as  $\mathcal{C}'_t$  has codimension bigger than or equal to two in  $\mathcal{X}_t$ . This implies that  $\mathcal{H}^{-1}(\mathbf{L}j_t^* \mathcal{H}^1(\mathcal{P})) = 0$  for any  $t \in T$ , hence  $\mathcal{H}^1(\mathcal{P})$  is flat over  $T$ .  $\square$

By the above lemma, the sheaf  $\mathcal{F}_t$  for  $t \neq 0$  is a flat deformation of  $\mathcal{F}_0$ . Since they have compact supports,  $\mathcal{F}_0$  and  $\mathcal{F}_t$  have the same Hilbert polynomials. It follows that, for a  $g$ -ample line bundle  $\mathcal{L}$  on  $\mathcal{X}$  with  $d := \deg(\mathcal{L}|_C) > 0$ , we have the equality

$$(23) \quad \chi(\mathcal{F}_0 \otimes (\mathcal{L} \boxtimes \mathcal{L})) = \chi(\mathcal{F}_t \otimes (\mathcal{L} \boxtimes \mathcal{L})).$$

By Lemma 2.3 and the Riemann-Roch theorem, we have

$$\begin{aligned} \chi(\mathcal{F}_0 \otimes (\mathcal{L} \boxtimes \mathcal{L})) &= \dim_{\mathbb{C}} A_{\text{con}} \cdot \chi(\mathcal{O}_C(-1) \otimes \mathcal{L})^2 \\ &= \dim_{\mathbb{C}} A_{\text{con}} \cdot d^2. \end{aligned}$$

By the definition of  $\mathcal{F}_t$  for  $t \neq 0$  in (20), we have

$$\begin{aligned} \chi(\mathcal{F}_t \otimes (\mathcal{L} \boxtimes \mathcal{L})) &= \sum_{j=1}^l \sum_{k=1}^{n_j} \chi(\mathcal{O}_{C_{j,k}}(-1) \otimes \mathcal{L})^2 \\ &= \sum_{j=1}^l j^2 \cdot n_j \cdot d^2. \end{aligned}$$

Here we have used the relation (15) for  $C' = C_{j,k}$ . Since  $d > 0$ , the equality (23) implies the desired equality for  $\text{wid}(C)$ .  $\square$

#### APPENDIX A. UNIQUENESS OF FOURIER-MUKAI KERNELS

Let  $Y$  be a quasi-projective complex variety, or a spectrum of a completion of a finitely generated  $\mathbb{C}$ -algebra at some maximum ideal. Suppose that  $f_i: X_i \rightarrow Y$  are projective morphisms for  $i = 1, 2$ , and  $X_i$  are regular schemes. Given an object

$$\mathcal{P} \in D^b \text{Coh}(X_1 \times X_2)$$

supported on  $X_1 \times_Y X_2$ , we have the Fourier-Mukai functor

$$\Phi_{X_1 \rightarrow X_2}^{\mathcal{P}}: D^b \text{Coh}(X_1) \rightarrow D^b \text{Coh}(X_2)$$

defined by

$$\Phi_{X_1 \rightarrow X_2}^{\mathcal{P}}(-) := \mathbf{R}p_{2*}(\mathbf{L}p_1^*(-) \overset{\mathbf{L}}{\otimes} \mathcal{P})$$

where  $p_i: X_1 \times X_2 \rightarrow X_i$  is the projection. The above functor preserves coherence since  $p_2|_{\text{Supp}(\mathcal{P})}$  is projective. For another regular scheme  $X_3$ , a projective morphism  $f_3: X_3 \rightarrow Y$  and an object  $\mathcal{Q} \in D^b \text{Coh}(X_2 \times X_3)$  supported on  $X_2 \times_Y X_3$ , we have

$$\Phi_{X_2 \rightarrow X_3}^{\mathcal{Q}} \circ \Phi_{X_1 \rightarrow X_2}^{\mathcal{P}} \cong \Phi_{X_1 \rightarrow X_3}^{\mathcal{Q} \circ \mathcal{P}}$$

where  $\mathcal{Q} \circ \mathcal{P}$  is defined by (cf. [Che02, Proposition 2.3])

$$\mathcal{Q} \circ \mathcal{P} := \mathbf{R}p_{13*}(p_{12}^* \mathcal{P} \overset{\mathbf{L}}{\otimes} p_{23}^* \mathcal{Q}).$$

Here  $p_{ij}: X_1 \times X_2 \times X_3 \rightarrow X_i \times X_j$  is the projection.

If  $Y = \text{Spec } \mathbb{C}$  and  $\Phi_{X_1 \rightarrow X_2}^{\mathcal{P}}$  is an equivalence, then Orlov [Orl97] showed that the kernel object  $\mathcal{P}$  is unique up to an isomorphism, i.e.  $\Phi_{X_1 \rightarrow X_2}^{\mathcal{P}} \cong \Phi_{X_1 \rightarrow X_2}^{\mathcal{Q}}$  implies  $\mathcal{P} \cong \mathcal{Q}$ . It should be well-known that the same claim holds without  $Y = \text{Spec } \mathbb{C}$  assumption, but as the author cannot find a reference we include a proof here.

**Lemma A.1.** *For  $\mathcal{P}, \mathcal{Q} \in D^b \text{Coh}(X_1 \times X_2)$  supported on  $X_1 \times_Y X_2$ , suppose that the following conditions hold:*

- *We have an isomorphism of functors  $\Phi_{X_1 \rightarrow X_2}^{\mathcal{P}} \cong \Phi_{X_1 \rightarrow X_2}^{\mathcal{Q}}$ .*
- *The functors  $\Phi_{X_1 \rightarrow X_2}^{\mathcal{P}}, \Phi_{X_1 \rightarrow X_2}^{\mathcal{Q}}$  are equivalences.*

*Then we have  $\mathcal{P} \cong \mathcal{Q}$ .*

*Proof.* Let  $\mathcal{Q}^*$  be the object of  $D^b \text{Coh}(X_1 \times X_2)$  given by

$$\mathcal{Q}^* := \mathbf{R}\mathcal{H}om_{X_1 \times X_2}(\mathcal{Q}, \mathcal{O}_{X_1 \times X_2}) \otimes p_1^* \omega_{X_1}[\dim X_1].$$

By the Grothendieck duality, the functor  $\Phi_{X_2 \rightarrow X_1}^{\mathcal{Q}^*}$  is the right adjoint of  $\Phi_{X_1 \rightarrow X_2}^{\mathcal{Q}}$ , hence an inverse of it. We have

$$\Phi_{X_2 \rightarrow X_1}^{\mathcal{Q}^*} \circ \Phi_{X_1 \rightarrow X_2}^{\mathcal{P}} \cong \Phi_{X_1 \rightarrow X_1}^{\mathcal{Q}^* \circ \mathcal{P}}$$

and it is isomorphic to the identity functor. Then  $\Phi_{X_1 \rightarrow X_1}^{\mathcal{Q}^* \circ \mathcal{P}}$  sends  $\mathcal{O}_x$  to  $\mathcal{O}_x$  for any  $x \in X_1$ , and  $\mathcal{O}_{X_1}$  to  $\mathcal{O}_{X_1}$ . Applying the argument of [Huy06, Corollary 5.23], it follows that  $\mathcal{Q}^* \circ \mathcal{P} \cong \mathcal{O}_{\Delta_{X_1}}$ . Similarly we have  $\mathcal{Q} \circ \mathcal{Q}^* \cong \mathcal{O}_{\Delta_{X_2}}$ . We obtain

$$\mathcal{P} \cong \mathcal{O}_{\Delta_{X_2}} \circ \mathcal{P} \cong \mathcal{Q} \circ \mathcal{Q}^* \circ \mathcal{P} \cong \mathcal{Q} \circ \mathcal{O}_{\Delta_{X_1}} \cong \mathcal{Q}$$

as desired.  $\square$

## APPENDIX B. CORRECTION ON FLOP-FLOP AUTOEQUIVALENCE

In this occasion, I would correct a wrong statement in [Tod07, Section 3] on the description of flop-flop autoequivalence. Let us consider the equivalence

$$(24) \quad \Phi_{X^\dagger \rightarrow X}^{\mathcal{O}_{X \times_Y X^\dagger}} \circ \Phi_{X \rightarrow X^\dagger}^{\mathcal{O}_{X \times_Y X^\dagger}} : D^b \text{Coh}(X) \xrightarrow{\sim} D^b \text{Coh}(X)$$

associated to the flop diagram (9). In [Tod07, Theorem 3.1], it was stated that if  $C$  is either a  $(-1, -1)$ -curve or a  $(0, -2)$ -curve, then the functor (24) is isomorphic to the generalized (commutative) twist functor  $T_{\mathcal{E}}$ . However this turns out to be wrong: the correct statement is that the equivalence (24) is an inverse of  $T_{\mathcal{E}}$ . Indeed the statement in [Tod07, Section 3] that the equivalence

$$(25) \quad \Phi_{X \rightarrow X^\dagger}^{\mathcal{O}_{X \times_Y X^\dagger}} : D^b \text{Coh}(X) \xrightarrow{\sim} D^b \text{Coh}(X^\dagger).$$

takes  $\mathcal{O}_C(-1)[1]$  to  $\mathcal{O}_{C^\dagger}(-1)$  was wrong: it should be corrected that (25) takes  $\mathcal{O}_C(-1)$  to  $\mathcal{O}_{C^\dagger}(-1)[1]$ . Then replacing  $T_{\mathcal{E}}$  with  $T_{\mathcal{E}}^{-1}$  in the proof of [Tod07, Theorem 3.1], we obtain the statement that (11) is isomorphic to  $T_{\mathcal{E}}^{-1}$ .

We explain why the above statement in [Tod07, Section 3] was wrong. In [Tod07, Section 3], I referred [Tod08, Ver 1, Lemma 5.1], which in turn referred [Bri02, (4.8)] that the equivalence (25) induces the equivalence

$$(26) \quad {}^{-1}\text{Per}(X/Y) \xrightarrow{\sim} {}^0\text{Per}(X^\dagger/Y).$$

(Here we have used the fact that the equivalence (25) coincides with the equivalence  $\Phi$  given in [Bri02, Section 4] by [Che02]). However (26) was not correct: it should be corrected that (25) induces the equivalence<sup>3</sup>

$$(27) \quad {}^0\text{Per}(X/Y) \xrightarrow{\sim} {}^{-1}\text{Per}(X^\dagger/Y).$$

Indeed let  $\mathcal{C}_X \subset \text{Coh}(X)$  be the category of sheaves  $F$  with  $\mathbf{R}f_*F = 0$ . Then [Bri02, (4.5)] shows that (25) takes  $\mathcal{C}_X$  to  $\mathcal{C}_{X^\dagger}[1]$ . On the other hand, as  ${}^p\text{Per}(X/Y)$  is the gluing of  $\text{Coh}(Y)$  and  $\mathcal{C}_X[-p]$  (not  $\mathcal{C}_X[p]$ ) by the definition, the equivalence (25) should reduce the perversity one. After correcting (26) as (27), the argument of [Tod08, Ver 1, Lemma 5.1] shows that (25) takes  $\mathcal{O}_C(-1)$  to  $\mathcal{O}_{C^\dagger}(-1)[1]$ .

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<sup>3</sup>In the notation of [Bri02, (4.8)], the equivalence  ${}^p\text{Per}(W/X) \cong {}^{p+1}\text{Per}(Y/X)$  should be corrected as  ${}^p\text{Per}(W/X) \cong {}^{p-1}\text{Per}(Y/X)$

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